

Chapter

Binomial Theorem



Topic-1: Binomial Theorem for a Positive Integral Index 'X', Expansion of Binomial, General Term, Coefficient of any Power of 'X'



1 MCQs with One Correct Answer

1. For $r = 0, 1, \dots, 10$, let A_r , B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1+x)^{10}$, [2010]

$(1+x)^{20}$ and $(1+x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to

- (a) $B_{10} - C_{10}$ (b) $A_{10}(B_{10}^2 - C_{10}A_{10})$
 (c) 0 (d) $C_{10} - B_{10}$
2. Coefficient of t^{24} in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is [2003S]
 (a) ${}^{12}C_6 + 3$ (b) ${}^{12}C_6 + 1$ (c) ${}^{12}C_6$ (d) ${}^{12}C_6 + 2$
3. In the binomial expansion of $(a-b)^n$, $n \geq 5$, the sum of the 5th and 6th terms is zero. Then a/b equals [2001S]
 (a) $(n-5)/6$ (b) $(n-4)/5$
 (c) $5/(n-4)$ (d) $6/(n-5)$

4. The coefficient of x^4 in $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is [1983 - 1 Mark]

- (a) $\frac{405}{256}$ (b) $\frac{504}{259}$
 (c) $\frac{450}{263}$ (d) none of these

5. Given positive integers $r > 1$, $n > 2$ and that the coefficient of $(3r)$ th and $(r+2)$ th terms in the binomial expansion of $(1+x)^{2n}$ are equal. Then [1983 - 1 Mark]

- (a) $n = 2r$ (c) $n = 2r + 1$
 (c) $n = 3r$ (d) none of these



2 Integer Value Answer/ Non-Negative Integer

6. Let a and b be two non-zero real numbers. If the coefficient of x^5 in the expansion of $\left(ax^2 + \frac{70}{27bx}\right)^4$ is equal to the

coefficient of x^{-5} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^7$, then

the value of $2b$ is

[Adv. 2023]

7. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2 + (1+x)^3 + \dots + (1+x)^{49} + (1+mx)^{50}$ is $(3n+1) {}^{51}C_3$ for some positive integer n . Then the value of n is [Adv. 2016]
8. The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14. Then $n =$ [Adv. 2013]



3 Numeric/ New Stem Based Questions

9. Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \dots + 10({}^{10}C_{10})^2$, where ${}^{10}C_r$, $r \in \{1, 2, \dots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430} X$ is _____. [Adv. 2018]



4 Fill in the Blanks

10. The sum of the rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is _____. [1997 - 2 Marks]
11. Let n be positive integer. If the coefficients of 2nd, 3rd, and 4th terms in the expansion of $(1+x)^n$ are in A.P., then the value of n is _____. [1994 - 2 Marks]
12. The larger of $99^{50} + 100^{50}$ and 101^{50} is _____. [1982 - 2 Marks]



6 MCQs with One or More than One Correct Answer

13. If $a_n = \sum_{r=0}^n \frac{1}{{}^nC_r}$, then $\sum_{r=0}^n \frac{r}{{}^nC_r}$ equals [1998 - 2 Marks]
 (a) $(n-1)a_n$ (b) na_n
 (c) $\frac{1}{2}na_n$ (d) None of the above



Topic-2: Middle Term, Greatest Term, Independent Term, Particular Term from end in Binomial Expansion, Greatest Binomial Coefficient



10 Subjective Problems

- Prove that $\sum_{r=1}^k (-3)^{r-1} {}^{3n}C_{2r-1} = 0$, where $k = (3n)/2$ and n is an even positive integer. [1993 - 5 Marks]
- Let $R = (5\sqrt{5} + 11)^{2n+1}$ and $f = R - [R]$, where $[]$ denotes the greatest integer function. Prove that $Rf = 4^{2n+1}$. [1988 - 5 Marks]



Topic-3: Properties of Binomial Coefficients, Number of Terms in the Expansion of $(x + y + z)^n$, Binomial Theorem for any Index, Multinomial Theorem, Infinite Series

- Coefficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$ is [Adv. 2014]
(a) 1051 (b) 1106 (c) 1113 (d) 1120

- The value of $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \dots + \binom{30}{20}\binom{30}{30}$ is where $\binom{n}{r} = {}^nC_r$ [2005S]

- (a) $\binom{30}{10}$ (b) $\binom{30}{15}$ (c) $\binom{60}{30}$ (d) $\binom{31}{10}$
- The sum $\sum_{i=0}^m \binom{10}{i} \binom{20}{m-i}$, (where $\binom{p}{q} = 0$ if $p < q$) is maximum when m is [2002S]
(a) 5 (b) 10 (c) 15 (d) 20

- For $2 \leq r \leq n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$ [2000S]
(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$
(c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$

- If in the expansion of $(1+x)^m(1-x)^n$, the coefficients of x and x^2 are 3 and -6 respectively, then m is [1999 - 2 Marks]
(a) 6 (b) 9 (c) 12 (d) 24

- The expression $\left[x + (x^3 - 1)^{\frac{1}{2}} \right]^5 + \left[x - (x^3 - 1)^{\frac{1}{2}} \right]^5$ is a polynomial of degree [1992 - 2 Marks]
(a) 5 (b) 6 (c) 7 (d) 8



3 Numeric/ New Stem Based Questions

- Suppose $\det \begin{bmatrix} \sum_{k=0}^n k & \sum_{k=0}^n {}^nC_k k^2 \\ \sum_{k=0}^n {}^nC_k k & \sum_{k=0}^n {}^nC_k k^3 \end{bmatrix} = 0$ holds for some positive integer n . The $\sum_{k=0}^n \frac{{}^nC_k}{k+1}$ equals ____ [Adv. 2019]



4 Fill in the Blanks

- The sum of the coefficients of the polynomial $(1 + x - 3x^2)^{2163}$ is [1982 - 2 Marks]



6 MCQs with One or More than One Correct Answer

- If C_r stands for nC_r , then the sum of the series $\frac{2\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!}{n!} [C_0^2 - 2C_1^2 + 3C_2^2 - \dots + (-1)^n (n+1)C_n^2]$, where n is an even positive integer, is equal to [1986 - 2 Marks]
(a) 0 (b) $(-1)^{n/2} (n+1)$
(c) $(-1)^{n/2} (n+2)$ (d) $(-1)^n n$



10 Subjective Problems

- Prove that [2003 - 2 Marks]
$$2^k \binom{n}{0} \binom{n}{k} - 2^{k-1} \binom{n}{1} \binom{n-1}{k-1} + 2^{k-2} \binom{n}{2} \binom{n-2}{k-2} - \dots + (-1)^k \binom{n}{k} \binom{n-k}{0} = \binom{n}{k}$$

11. For any positive integer m, n (with $n \geq m$), let $\binom{n}{m} = {}^nC_m$.

Prove that $\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \dots + \binom{m}{m} = \binom{n+1}{m+2}$.

Hence or otherwise, prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \dots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}.$$

[2000 - 6 Marks]

12. Let n be a positive integer and

$$(1+x+x^2)^n = a_0 + a_1x + \dots + a_{2n}x^{2n}$$

$$\text{Show that } a_0^2 - a_1^2 + a_2^2 - \dots + a_{2n}^2 = a_n$$

[1994 - 5 Marks]

13. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all

$$k \geq n, \text{ then show that } b_n = {}^{2n+1}C_{n+1} \quad [1992 - 6 \text{ Marks}]$$

14. Prove that

$$C_0 - 2^2 C_1 + 3^2 C_2 - \dots + (-1)^n (n+1)^2 C_n = 0,$$

$$n > 2, \text{ where } C_r = {}^nC_r.$$

[1989 - 5 Marks]

15. Given $s_n = 1 + q + q^2 + \dots + q^n$;

$$S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + \dots + \left(\frac{q+1}{2}\right)^n, q \neq 1 \text{ Prove}$$

$$\text{that } {}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + \dots + {}^{n+1}C_{n+1} s_n = 2^n S_n$$

[1984 - 4 Marks]

16. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ then show that the sum of the products of the C_i 's taken two at a time,

$$\text{represented by } \sum_{0 \leq i < j \leq n} C_i C_j \text{ is equal to}$$

$$2^{2n-1} - \frac{(2n)!}{2(n!)^2}$$

[1983 - 3 Marks]

17. Given that

$$C_1 + 2C_2x + 3C_3x^2 + \dots + 2nC_{2n}x^{2n-1} = 2n(1+x)^{2n-1}$$

$$\text{where } C_r = \frac{(2n)!}{r!(2n-r)!} \quad r=0, 1, 2, \dots, 2n$$

Prove that

$$C_1^2 - 2C_2^2 + 3C_3^2 - \dots - 2nC_{2n}^2 = (-1)^n n C_n.$$

[1979]



Answer Key

Topic-1 : Binomial Theorem for a Positive Integral Index 'x', Expansion of Binomial, General

Term, Coefficient of any Power of 'x'

1. (d) 2. (d) 3. (b) 4. (a) 5. (a) 6. (3) 7. (5) 8. (6) 9. (646) 10. (41)
11. (7 or 2) 12. $(101)^{50}$ 13. (c)

Topic-3 : Properties of Binomial Coefficients, Number of Terms in the Expansion of $(x+y+z)^n$,

Binomial Theorem for any Index, Multinomial Theorem, Infinite Series

1. (c) 2. (a) 3. (c) 4. (d) 5. (c) 6. (c) 7. (6.20) 8. (-1) 9. (c)

Hints & Solutions



Topic-1: Binomial Theorem for a Positive Integral Index 'x', Expansion of Binomial, General Term, Coefficient of any Power of 'x'

1. (d) Clearly $A_r = {}^{10}C_r$, $B_r = {}^{20}C_r$, $C_r = {}^{30}C_r$

$$\begin{aligned} \text{Now } \sum_{r=1}^{10} A_r (B_{10} B_r - C_{10} A_r) \\ = \sum_{r=1}^{10} {}^{10}C_r \left({}^{20}C_{10} {}^{20}C_r - {}^{30}C_{10} {}^{10}C_r \right) \\ = {}^{20}C_{10} \sum_{r=1}^{10} {}^{10}C_r {}^{20}C_r - {}^{30}C_{10} \sum_{r=1}^{10} {}^{10}C_r \times {}^{10}C_r \\ = {}^{20}C_{10} \left({}^{10}C_1 {}^{20}C_1 + {}^{10}C_2 {}^{20}C_2 + \dots + {}^{10}C_{10} {}^{20}C_{10} \right) \\ - {}^{30}C_{10} \left({}^{10}C_1 \times {}^{10}C_1 + {}^{10}C_2 \times {}^{10}C_2 + \dots + {}^{10}C_{10} {}^{10}C_{10} \right) \dots (i) \end{aligned}$$

Now on expanding $(1+x)^{10}$ and $(1+x)^{20}$ and comparing the coefficients of x^{20} in their product on both sides, we get

$$\begin{aligned} {}^{10}C_0 {}^{20}C_0 + {}^{10}C_1 {}^{20}C_1 + {}^{10}C_2 {}^{20}C_2 + \dots + {}^{10}C_{10} {}^{20}C_{10} \\ = \text{Coeff. of } x^{20} \text{ in } (1+x)^{30} = {}^{30}C_{20} = {}^{30}C_{10} \\ \therefore {}^{10}C_1 {}^{20}C_1 + {}^{10}C_2 {}^{20}C_2 + \dots + {}^{10}C_{10} {}^{20}C_{10} = {}^{30}C_{10} - 1 \dots (ii) \end{aligned}$$

Again on expanding $(1+x)^{10}$ and $(x+1)^{10}$ and comparing the coefficients of x^{10} in their product on both sides, we get

$$\begin{aligned} \therefore ({}^{10}C_0)^2 + ({}^{10}C_1)^2 + ({}^{10}C_2)^2 + \dots + ({}^{10}C_{10})^2 \\ = \text{Coeff. of } x^{10} \text{ in } (1+x)^{20} = {}^{20}C_{10} \\ \therefore ({}^{10}C_1)^2 + ({}^{10}C_2)^2 + \dots + ({}^{10}C_{10})^2 = {}^{20}C_{10} - 1 \dots (iii) \end{aligned}$$

Now, from equations (i), (ii) and (iii), we get

$$\begin{aligned} \text{Required value} &= {}^{20}C_{10} \left({}^{30}C_{10} - 1 \right) - {}^{30}C_{10} \left({}^{20}C_{10} - 1 \right) \\ &= {}^{30}C_{10} - {}^{20}C_{10} = C_{10} - B_{10} \end{aligned}$$

2. (d) $(1+t^2)^{12} (1+t^{12}) (1+t^{24})$
 $= (1+t^{12} + t^{24} + t^{36}) (1+t^2)^{12}$
 $\therefore$ Coeff. of $t^{24} = 1 \times \text{Coeff. of } t^{24} \text{ in } (1+t^2)^{12} + 1 \times$
 $\text{Coeff. of } t^{12} \text{ in } (1+t^2)^{12} + 1 \times \text{constant term in } (1+t^2)^{12}$
 $= {}^{12}C_{12} + {}^{12}C_6 + {}^{12}C_0 = 1 + {}^{12}C_6 + 1 = {}^{12}C_6 + 2$
3. (b) In binomial expansion $(a-b)^n$, $n \geq 5$;
 $T_5 + T_6 = 0$
 $\Rightarrow {}^nC_4 a^{n-4} b^4 - {}^nC_5 a^{n-5} b^5 = 0$
 $\Rightarrow \frac{{}^nC_4}{{}^nC_5} \cdot \frac{a}{b} = 1 \Rightarrow \frac{5}{n-4} \cdot \frac{a}{b} = 1 \Rightarrow \frac{a}{b} = \frac{n-4}{5}$
4. (a) General term in the expansion $\left(\frac{x}{2} - \frac{3}{x^2} \right)^{10}$ is

$$T_{r+1} = {}^{10}C_r \left(\frac{x}{2} \right)^{10-r} \left(\frac{-3}{x^2} \right)^r = {}^{10}C_r x^{10-3r} \frac{(-1)^r 3^r}{2^{10-r}}$$

To find coeff of x^4 , put $10-3r=4 \Rightarrow r=2$

$$\therefore \text{Coeff of } x^4 = {}^{10}C_2 \frac{(-1)^2 3^2}{2^8} = \frac{405}{256}$$

5. (a) Given : r and n are positive integers such that $r > 1$, $n > 2$

Also, in the expansion of $(1+x)^{2n}$

Coeff. of $(3r)^{\text{th}}$ term = Coeff. of $(r+2)^{\text{th}}$ term

$$\Rightarrow {}^{2n}C_{3r-1} = {}^{2n}C_{r+1} \Rightarrow 3r-1 = r+1 \text{ or } 3r-1+r+1 = 2n$$

$$\left[\because \text{If } {}^nC_p = {}^nC_q, \text{ then } p = q \text{ or } p+q = n \right]$$

$$\Rightarrow r = 1 \text{ or } 2r = n$$

$$\text{But } r > 1 \therefore n = 2r$$

6. (3) Given expansion $\left(ax^2 + \frac{70}{27bx} \right)^4$

$$\begin{aligned} T_{r+1} &= {}^4C_r (ax^2)^{4-r} \left(\frac{70}{27bx} \right)^r \\ &= {}^4C_r a^{4-r} \left(\frac{70}{27b} \right)^r x^{8-3r} \end{aligned}$$

$$\text{Here, } 8-3r=5 \Rightarrow r=1$$

$$\text{So, coefficient of } x^5 = {}^4C_1 a^3 \cdot \frac{70}{27b}$$

$$\text{For expansion } \left(ax - \frac{1}{bx^2} \right)^7$$

$$T_{r+1} = {}^7C_r (ax)^{7-r} \left(\frac{-1}{bx^2} \right)^r = {}^7C_r a^{7-r} \left(\frac{-1}{b} \right)^r x^{7-3r}$$

$$\text{Here, } 7-3r=-5 \Rightarrow r=4$$

$$\text{So, coefficient of } x^{-5} = {}^7C_4 a^3 \left(\frac{-1}{b} \right)^3$$

$$\text{A.T.Q, } {}^4C_1 a^3 \frac{70}{27b} = {}^7C_4 a^3 \cdot \frac{-1}{b^3}$$

$$\Rightarrow b = \frac{3}{2} \Rightarrow 2b = 3.$$

7. (5) $(1+x)^2 + (1+x)^3 + \dots + (1+x)^{49} + (1+mx)^{50}$

$$= (1+x)^2 \left[\frac{(1+x)^{48} - 1}{(1+x) - 1} \right] + (1+mx)^{50}$$

$$= \frac{1}{x} \left[(1+x)^{50} - (1+x)^2 \right] + (1+mx)^{50}$$

Coeff. of x^2 in the above expansion

$$= \text{Coeff. of } x^3 \text{ in } (1+x)^{50} + \text{Coeff. of } x^2 \text{ in } (1+mx)^{50}$$

$$= {}^{50}C_3 + {}^{50}C_2 m^2$$

$$\therefore (3n+1) {}^{51}C_3 = {}^{50}C_3 + {}^{50}C_2 m^2$$

$$\Rightarrow (3n+1) = \frac{{}^{50}C_3}{{}^{51}C_3} + \frac{{}^{50}C_2}{{}^{51}C_3} m^2$$

$$\Rightarrow 3n+1 = \frac{16}{17} + \frac{1}{17} m^2 \Rightarrow n = \frac{m^2-1}{51}$$

$\therefore$ Least positive integer m for which n is an integer is $m=16$ and then $n=5$

8. (6) Let the coefficients of three consecutive terms of $(1+x)^{n+5}$ be ${}^{n+5}C_{r-1}$, ${}^{n+5}C_r$, ${}^{n+5}C_{r+1}$, then we have

$${}^{n+5}C_{r-1} : {}^{n+5}C_r : {}^{n+5}C_{r+1} = 5 : 10 : 14$$

$$\frac{{}^{n+5}C_{r-1}}{{}^{n+5}C_r} = \frac{5}{10} \Rightarrow \frac{r}{n+6-r} = \frac{1}{2}$$

$$\Rightarrow n-3r+6=0 \quad \dots(i)$$

$$\text{Also } \frac{{}^{n+5}C_r}{{}^{n+5}C_{r+1}} = \frac{10}{14} \Rightarrow \frac{r+1}{n-r+5} = \frac{5}{7}$$

$$\Rightarrow 5n-12r+18=0 \quad (ii)$$

Solving (i) and (ii), we get $n=6$.

9. (646) $\sum_{r=0}^n r({}^nC_r)^2 = n \sum_{r=0}^n {}^nC_r {}^{n-1}C_{r-1}$

$$= n \sum_{r=1}^n {}^nC_{n-r} {}^{n-1}C_{r-1} = n {}^{2n-1}C_{n-1}$$

Now, $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \dots + 10({}^{10}C_{10})^2$

$$= \sum_{n=0}^{10} r({}^{10}C_r)^2 = 10 {}^{19}C_9 \therefore \frac{X}{1430} = \frac{1}{143} {}^{19}C_9 = 646$$

10. Given expression : $(\sqrt{2} + 3^{1/5})^{10}$

$$\therefore T_{r+1} = {}^{10}C_r (\sqrt{2})^{10-r} \cdot (3^{1/5})^r \cdot (0 \leq r \leq 10)$$

$$= \frac{10!}{r!(10-r)!} \cdot 2^{5-r/2} \cdot 3^{r/5}$$

T_{r+1} will be rational if $2^{5-r/2}$ and $3^{r/5}$ are rational numbers.

$$\Rightarrow 5 - \frac{r}{2} \text{ and } \frac{r}{5} \text{ are integers.}$$

$$\Rightarrow r=0 \text{ and } r=10 \Rightarrow T_1 \text{ and } T_{11} \text{ are rational terms.}$$

$$\text{Now, } T_1 + T_{11} = {}^{10}C_0 2^5 \cdot 0.3^0 + {}^{10}C_{10} 2^{5-5} \cdot 3^2$$

$$= 1.32.1 + 1.1.9 = 32 + 9 = 41$$

11. We know that for a positive integer n

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$$

Since coefficients of 2^{nd} , 3^{rd} and 4^{th} terms are in A.P.

$$\therefore {}^nC_1, {}^nC_2, {}^nC_3 \text{ are in A.P.}$$

$$\Rightarrow 2 \cdot {}^nC_2 = {}^nC_1 + {}^nC_3$$

$$\Rightarrow 2 \times \frac{n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{3!}$$

$$\Rightarrow n-1 = 1 + \frac{n^2-3n+2}{6} \Rightarrow n^2-9n+14=0$$

$$\Rightarrow (n-7)(n-2)=0 \Rightarrow n=7 \text{ or } 2$$

But for the existence of 4^{th} term, $n=7$.

12. $(101)^{50} - \{(99)^{50} + (100)^{50}\}$

$$= (100+1)^{50} - (100-1)^{50} - (100)^{50}$$

$$= (100)^{50} [(1+0.01)^{50} - (1-0.01)^{50} - 1]$$

$$= (100)^{50} [2({}^{50}C_1(0.01) + {}^{50}C_3(0.01)^3 + \dots) - 1]$$

$$= (100)^{50} \left[2 \times 50 \times \frac{1}{100} + 2({}^{50}C_3(0.01)^3 + \dots) - 1 \right]$$

$$= (100)^{50} [2({}^{50}C_3(0.01)^3 + \dots)] > 0$$

$$\therefore (101)^{50} > (99)^{50} + (100)^{50} \therefore (101)^{50} \text{ is greater.}$$

13. (c) Let $b = \sum_{r=0}^n \frac{r}{{}^nC_r} = \sum_{r=0}^n \frac{n-(n-r)}{{}^nC_r}$

$$= na_n - \sum_{r=0}^n \frac{n-r}{{}^nC_{n-r}} = na_n - b \quad [\because {}^nC_r = {}^nC_{n-r}]$$

$$\Rightarrow 2b = na_n \Rightarrow b = \frac{n}{2} a_n$$

Topic-2: Middle Term, Greatest Term, Independent Term, Particular Term from end in Binomial Expansion, Greatest Binomial Coefficients

1. Given $k = \frac{3n}{2}$, where n is an even positive integer. Now let $n = 2m$ so that $k = 3m$

$$\text{Since } \sum_{r=1}^k (-3)^{r-1} {}^{3n}C_{2r-1} = \sum_{r=1}^{3m} (-3)^{r-1} {}^{6m}C_{2r-1}$$

$$= {}^{6m}C_1 - 3 {}^{6m}C_3 + 3^2 {}^{6m}C_5 - 3^3 {}^{6m}C_7 \dots$$

Consider $(\cos \theta - i \sin \theta)^{6m}$

$$= \cos^{6m} \theta - {}^{6m}C_1 \cos^{6m-1} \theta i \sin \theta$$

$$+ {}^{6m}C_2 \cos^{6m-2} \theta i^2 \sin^2 \theta - {}^{6m}C_3 \cos^{6m-3} \theta i^3 \sin^3 \theta \dots$$

$$\text{L.H.S.} = \cos^{6m} \theta - i \sin^{6m} \theta$$

Equating imaginary parts we get $\sin^{6m} \theta$

$$= {}^{6m}C_1 \cos^{6m-1} \theta \sin \theta - {}^{6m}C_3 \cos^{6m-3} \theta \sin^3 \theta + \dots$$

$$\text{Now put } \theta = \frac{\pi}{3} \therefore 6\theta = 2\pi$$

$$\text{or } \sin^{6m} \theta = \sin^{2m} 2\pi = 0$$

$$\therefore 0 = {}^{6m}C_1 \frac{\sqrt{3}}{2} \left(\frac{1}{2}\right)^{6m-1} \dots - {}^{6m}C_3 \left(\frac{\sqrt{3}}{2}\right)^3 \left(\frac{1}{2}\right)^{6m-2} + \dots$$

$$\text{or } 0 = \frac{\sqrt{3}}{2(6m)} [{}^{6m}C_1 - 3 {}^{6m}C_3 + 3^2 {}^{6m}C_5 + \dots]$$

$$\therefore {}^{6m}C_1 - 3 {}^{6m}C_3 + 3^2 {}^{6m}C_5 \dots = 0$$

$$\Rightarrow \sum_{r=1}^k (-3)^{r-1} {}^{3n}C_{2r-1} = 0.$$

2. $5\sqrt{5} - 11 = \frac{4}{5\sqrt{5} + 11} < 1 \therefore 0 < 5\sqrt{5} - 11 < 1$

$$\Rightarrow 0 < (5\sqrt{5} - 11)^{2n+1} < 1 \text{ for every positive integer } n.$$

$$\text{Also } (5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$$

$$= 2[{}^{2n+1}C_1(5\sqrt{5})^{2n} \cdot 11 + {}^{2n+1}C_3(5\sqrt{5})^{2n-2} \cdot 11^3 + \dots + {}^{2n+1}C_{2n+1} 11^{2n+1}]$$

$$= 2[{}^{2n+1}C_1(125)^n \cdot 11 + {}^{2n+1}C_3(125)^{n-1} \cdot 11^3 + \dots + {}^{2n+1}C_{2n+1}11^{2n+1}]$$

$$= 2k \quad \dots(i)$$

where k is some positive integer.

$$\text{Let } F = (5\sqrt{5} - 11)^{2n+1}$$

Then equation (i) becomes

$$R - F = 2k \quad \left[\because R = (5\sqrt{5} + 11)^{2n+1} \right]$$

$$\Rightarrow [R] + R - [R] - F = 2k \Rightarrow [R] + f - F = 2k$$

$$\Rightarrow f - F = 2k - [R] \Rightarrow f - F \text{ is an integer.}$$

$$\text{But } 0 \leq f < 1 \text{ and } 0 < F < 1, \therefore -1 < f - F < 1$$

$$\because f - F \text{ is an integer, we must have } f - F = 0$$

$$\Rightarrow f = F.$$

$$\text{Now, } Rf = RF = (5\sqrt{5} + 11)^{2n+1} (5\sqrt{5} - 11)^{2n+1}$$

$$= [(5\sqrt{5})^2 - 12]^{2n+1} = 4^{2n+1}$$

Topic-3: Properties of Binomial Coefficients, Number of Terms in the Expansion of $(x+y+z)^n$

Binomial Theorem for any Index, Multinomial Theorem, Infinite Series

1. (c) Coeff. of x^{11} in exp. of $(1+x^2)^4 (1+x^3)^7 (1+x^4)^{12}$
 $= [\text{Coeff. of } x^a \text{ in } (1+x^2)^4] \times [\text{Coeff. of } x^b \text{ in } (1+x^3)^7] \times [\text{Coeff. of } x^c \text{ in } (1+x^4)^{12}]$

$$\text{Such that } a + b + c = 11$$

$$\text{Here } a = 2m, b = 3n, c = 4p$$

$$\therefore 2m + 3n + 4p = 11$$

$$\text{Case I : } m = 0, n = 1, p = 2$$

$$\text{Case II : } m = 1, n = 3, p = 0$$

$$\text{Case III : } m = 2, n = 1, p = 1$$

$$\text{Case IV : } m = 4, n = 1, p = 0$$

$$\therefore \text{Required coefficient.}$$

$$= {}^4C_0 \times {}^7C_1 \times {}^{12}C_2 + {}^4C_1 \times {}^7C_3 \times {}^{12}C_0$$

$$+ {}^4C_2 \times {}^7C_1 \times {}^{12}C_4 + {}^4C_4 \times {}^7C_1 \times {}^{12}C_0$$

$$= 462 + 140 + 504 + 7 = 1113$$

2. (a) To find
 ${}^{30}C_0 {}^{30}C_{10} - {}^{30}C_1 {}^{30}C_{11} + {}^{30}C_2 {}^{30}C_{12} - \dots + {}^{30}C_{20} {}^{30}C_{30}$

$$\therefore (1+x)^{30} = {}^{30}C_0 + {}^{30}C_1 x + {}^{30}C_2 x^2 + \dots + {}^{30}C_{20} x^{20} + \dots + {}^{30}C_{30} x^{30} \quad \dots(i)$$

$$\text{and } (x-1)^{30} = {}^{30}C_0 x^{30} - {}^{30}C_1 x^{29} + \dots + {}^{30}C_{10} x^{20} - {}^{30}C_{11} x^{19} + {}^{30}C_{12} x^{18} - \dots + {}^{30}C_{30} x^0 \quad \dots(ii)$$

On multiplying equations (i) and (ii), we get

$$(x^2-1)^{30} = (\quad) \times (\quad)$$

Equating the coefficients of x^{20} on both sides, we get

$${}^{30}C_{10} = {}^{30}C_0 {}^{30}C_{10} - {}^{30}C_1 {}^{30}C_{11} + {}^{30}C_2 {}^{30}C_{12} - \dots + {}^{30}C_{20} {}^{30}C_{30}$$

$$\therefore \text{Required value} = {}^{30}C_{10}$$

3. (c) $\sum_{i=0}^n {}^{10}C_i {}^{20}C_{m-i} = {}^{10}C_0 {}^{20}C_m + {}^{10}C_1 {}^{20}C_{m-1} + {}^{10}C_2 {}^{20}C_{m-2} + \dots + {}^{10}C_m {}^{20}C_0$

$$= \text{Coeff. of } x^m \text{ in the expansion of product } (1+x)^{10} (1+x)^{20}$$

$$= \text{Coeff. of } x^m \text{ in the expansion of } (1+x)^{30} = {}^{30}C_m$$

$$\sum_{i=0}^n {}^{10}C_i {}^{20}C_{m-i} \text{ will be maximum, if } {}^{30}C_m \text{ will be maximum.}$$

$$\text{Clearly, } {}^{30}C_m \text{ will be maximum when } m = \frac{30}{2} = 15$$

$$\therefore \text{Max. } ({}^nC_r) = \begin{cases} {}^nC_{n/2} \text{ if } n \text{ is even} \\ {}^nC_{\frac{n+1}{2}} \text{ if } n \text{ is odd} \end{cases}$$

4. (d) $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$
 $= \left[\binom{n}{r} + \binom{n}{r-1} \right] + \left[\binom{n}{r-1} + \binom{n}{r-2} \right]$
 $= \left[\text{Here } \binom{n}{r}, \binom{n}{r-1} \text{ and } \binom{n}{r-2} \right]$
 $= \left[\text{represent } {}^nC_r, {}^nC_{r-1} \text{ and } {}^nC_{r-2} \right]$
 $= \binom{n+1}{r} + \binom{n+1}{r-1} = \binom{n+2}{r} \quad [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

5. (c) $(1+x)^m (1-x)^n$
 $= \left[1 + mx + \frac{m(m-1)}{2!} x^2 + \dots \right] \left[1 - nx + \frac{n(n-1)}{2!} x^2 - \dots \right]$
 $= 1 + (m-n)x + \left[\frac{m(m-1)}{2} + \frac{n(n-1)}{2} - mn \right] x^2 + \dots$
 Given, $m-n=3 \quad \dots(i)$

$$\text{and } \frac{1}{2} m(m-1) + \frac{1}{2} n(n-1) - mn = -6$$

$$\Rightarrow m^2 + n^2 - 2mn - (m+n) = -12$$

$$\Rightarrow (m-n)^2 - (m+n) = -12$$

$$\Rightarrow m+n = 9+12 = 21 \quad \dots(ii)$$

$$\text{From (i) and (ii), we get } m = 12$$

6. (c) Given expression :

$$(x + \sqrt{x^3-1})^5 + (x - \sqrt{x^3-1})^5$$

We know that using binomial theorem,

$$(x+a)^n + (x-a)^n = 2[{}^nC_0 x^n + {}^nC_2 x^{n-2} a^2 + \dots + {}^nC_n x^{n-n} a^n + \dots]$$

$\therefore$ The given expression

$$= 2[{}^5C_0 x^5 + {}^5C_2 x^3 (x^3-1) + {}^5C_4 x (x^3-1)^2]$$

Since maximum power of x involved in the expansion is 7. Also only +ve integral powers of x are involved in the expansion, therefore given expression is a polynomial of degree 7.

7. (6.20) Here $\sum_{k=0}^n k = \frac{n(n+1)}{2}$

$$\sum_{k=0}^n {}^nC_k k^2 = \sum_{k=1}^n \frac{n}{k} \cdot {}^{n-1}C_{k-1} \cdot k^2 = \sum_{k=1}^n n \cdot {}^{n-1}C_{k-1} \cdot k$$

$$= n \sum_{k=1}^n {}^{n-1}C_{k-1} (k-1+1)$$

$$\begin{aligned}
 &= n \left[\sum_{k=2}^n \frac{n-1}{k-1} {}^{n-2}C_{k-2} (k-1) + \sum_{k=1}^n {}^{n-1}C_{k-1} \right] \\
 &= n(n-1)2^{n-2} + n \times 2^{n-1} \\
 \sum_{k=0}^n {}^nC_k \times k &= \sum_{k=1}^n \frac{n}{k} {}^{n-1}C_{k-1} \times k = n \times 2^{n-1} \\
 \text{and } \sum_{k=0}^n {}^nC_k 3^k &= 4^n \\
 \therefore \det \begin{bmatrix} \sum_{k=0}^n k & \sum_{k=0}^n {}^nC_k k^2 \\ \sum_{k=0}^n {}^nC_k k & \sum_{k=0}^n {}^nC_k 3^k \end{bmatrix} &= 0 \\
 \Rightarrow \begin{vmatrix} \frac{n(n+1)}{2} & n(n-1)2^{n-2} + n \times 2^{n-1} \\ n \times 2^{n-1} & 4^n \end{vmatrix} &= 0 \\
 \Rightarrow n(n+1) \times 2^{2n-1} - n^2 [(n-1)2^{2n-3} + 2^{2n-2}] &= 0 \\
 \Rightarrow 2^{2n-3} \times n [4(n+1) - n(n-1+2)] &= 0 \\
 \Rightarrow 2^{2n-3} \times n [4n+4 - n^2 - n] &= 0 \\
 \Rightarrow n^2 - 3n - 4 = 0 \Rightarrow n = 4 \\
 \therefore \sum_{k=0}^n \frac{{}^nC_k}{k+1} &= \sum_{k=0}^4 \frac{{}^nC_k}{k+1} = {}^4C_0 + \frac{{}^4C_1}{2} + \frac{{}^4C_2}{3} + \frac{{}^4C_3}{4} + \frac{{}^4C_4}{5} \\
 &= 1 + 2 + 2 + 1 + \frac{1}{5} = 6.20
 \end{aligned}$$

8. On putting $x = 1$ in the expansion of $(1+x-3x^2)^{2163}$
 $= A_0 + A_1x + A_2x^2 + \dots$, we will get the sum of coefficients of
given polynomial, which clearly comes to be -1 .

9. (c) $\therefore n$ is even, let $n = 2m$ then

$$\begin{aligned}
 \text{LHL} = S &= \frac{2.m!m!}{(2m)!} [C_0^2 - 2C_1^2 + 3C_2^2 - \dots \\
 &+ (-1)^{2m} (2m+1) C_{2m}^2 \dots] \quad \dots(i) \\
 &= \frac{2.m!m!}{(2m)!} [C_{2m}^2 - 2C_{2m-1}^2 + 3C_{2m-2}^2 - \dots \\
 &+ (-1)^{2m} (2m+1) C_0^2 \dots] \quad [\text{using } C_r = C_{n-r}]
 \end{aligned}$$

$$\Rightarrow S = \frac{2.m!m!}{(2m)!} [(2m+1) C_0^2 - 2m C_1^2 + (2m-1) C_2^2 - \dots - 2C_{2m-1}^2 + C_{2m}^2] \quad \dots(ii)$$

On adding (i) and (ii),

$$2S = 2 \frac{m!m!}{(2m)!} [2m+2] [C_0^2 - C_1^2 + C_2^2 + \dots + C_{2m}^2]$$

Now keeping in mind that if n is even, then

$$C_0^2 - C_1^2 + C_2^2 - \dots + C_n^2 = (-1)^{n/2} {}^nC_{n/2}$$

$\therefore$ we get

$$\begin{aligned}
 S &= \frac{m!m!}{(2m)!} (2m+2) [(-1)^{m/2} {}^nC_{n/2}] \\
 &= \frac{m!m!}{(2m)!} \left(2 \cdot \frac{n}{2} + 2 \right) (-1)^m \frac{(2m)!}{m!m!} \\
 &= (n+2)(-1)^{n/2} = (-1)^{n/2} (n+2)
 \end{aligned}$$

10. To show that

$$2^k \cdot {}^nC_0 \cdot {}^nC_k - 2^{k-1} \cdot {}^nC_1 \cdot {}^{n-1}C_{k-1} + 2^{k-2} \cdot {}^nC_2 \cdot {}^{n-2}C_{k-2} - \dots + (-1)^k {}^nC_k \cdot {}^{n-k}C_0 = {}^nC_k$$

Taking LHS

$$\begin{aligned}
 &2^k \cdot {}^nC_0 \cdot {}^nC_k - 2^{k-1} \cdot {}^nC_1 \cdot {}^{n-1}C_{k-1} + \dots + (-1)^k {}^nC_k \cdot {}^{n-k}C_0 \\
 &= \sum_{r=0}^k (-1)^r \cdot 2^{k-r} \cdot {}^nC_r \cdot {}^{n-r}C_{k-r} \\
 &= \sum_{r=0}^k (-1)^r 2^{k-r} \cdot \frac{n!}{r!(n-r)!} \cdot \frac{(n-r)!}{(k-r)!(n-k)!} \\
 &= \sum_{r=0}^k (-1)^r \cdot 2^{k-r} \cdot \frac{n!}{(n-k)! \cdot k!} \cdot \frac{k!}{r!(k-r)!} \\
 &= \sum_{r=0}^k (-1)^r \cdot 2^{k-r} \cdot {}^nC_k \cdot {}^kC_r = 2^k \cdot {}^nC_k \left\{ \sum_{r=0}^k (-1)^r \cdot \frac{1}{2^r} \cdot {}^kC_r \right\} \\
 &= 2^k \cdot {}^nC_k \left(1 - \frac{1}{2} \right)^k = {}^nC_k = \text{R.H.S.}
 \end{aligned}$$

11. Given that for positive integers m and n such that $n \geq m$, then to prove that

$$\begin{aligned}
 &{}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \dots + {}^mC_m = {}^{n+1}C_{m+1} \\
 \text{L.H.S. } &{}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + \dots + {}^{n-1}C_m + {}^nC_m \quad [\text{writing L.H.S. in reverse order}] \\
 &= ({}^{m+1}C_{m+1} + {}^{m+1}C_m) + {}^{m+2}C_m + \dots + {}^{n-1}C_m + {}^nC_m \\
 &\quad [\because {}^mC_m = {}^{m+1}C_{m+1}] \\
 &= ({}^{m+2}C_{m+1} + {}^{m+2}C_m) + {}^{m+3}C_m + \dots + {}^nC_m \\
 &\quad [\because {}^{m+1}C_{m+1} + {}^nC_r = {}^{n+1}C_{r+1}] \\
 &= {}^{m+3}C_{m+1} + {}^{m+3}C_m + \dots + {}^nC_m \\
 &\text{Combining in the same way we get} \\
 &= {}^nC_{m+1} + {}^nC_m = {}^{n+1}C_{m+1} = \text{R.H.S.}
 \end{aligned}$$

Again we have to prove

$${}^nC_m + 2 \cdot {}^{n-1}C_m + 3 \cdot {}^{n-2}C_m + \dots + (n-m+1) {}^mC_m = {}^{n+2}C_{m+2}$$

$$\begin{aligned}
 &= [{}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \dots + {}^mC_m] + [{}^{n-1}C_m + {}^{n-2}C_m + \dots + {}^mC_m] + \dots + [{}^mC_m] \\
 &\quad [n-m+1 \text{ bracketed terms}] \\
 &= {}^{n+1}C_{m+1} + {}^nC_{m+1} + {}^{n-1}C_{m+1} + \dots + {}^mC_{m+1} \quad [\text{using previous result.}]
 \end{aligned}$$

$$= {}^{n+2}C_{m+2}$$

[Replacing n by $n+1$ and m by $m+1$ in the previous result.]

= R.H.S.

12. Given : $(1+x+x^2)^n = a_0 + a_1x + \dots + a_{2n}x^{2n}$ (i)

where n is a +ve integer.

On replacing x by $-\frac{1}{x}$ in equation (i), we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2} \right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + \dots + \frac{a_{2n}}{x^{2n}} \quad \dots(ii)$$

Multiplying equation (i) and (ii) :

$$\frac{(1+x+x^2)^n (x^2-x+1)^n}{x^{2n}}$$

$$= (a_0 + a_1x + \dots + a_{2n}x^{2n}) \left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_{2n}}{x^{2n}} \right)$$

Equating the constant terms on both sides we get

$$a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + a_{2n}^2 = \text{constant term in the}$$

$$\text{expansion of } \frac{[(1+x+x^2)(1-x+x^2)]^n}{x^{2n}}$$

= Coeff. of x^{2n} in the expansion of $(1+x^2+x^4)^n$

But replacing x by x^2 in equation (i), we have

$$(1+x^2+x^4)^n = a_0 + a_1x^2 + \dots + a_{2n}(x^2)^{2n}$$

$\therefore$ Coeff. of $x^{2n} = a_n$

$$\therefore a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + a_{2n}^2 = a_n$$

$$13. \text{ Given : } \sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r \quad \dots(i)$$

and $a_k = 1, \forall k \geq n$

To prove : $b_n = {}^{2n+1}C_{n+1}$

In the given equation (i) let us put $x-3 = y$

$$\Rightarrow x-2 = y+1$$

$$\therefore \sum_{r=0}^{2n} a_r (1+y)^r = \sum_{r=0}^{2n} b_r (y)^r \quad [\text{From (i)}]$$

$$\Rightarrow a_0 + a_1(1+y) + \dots + a_{n-1}(1+y)^{n-1} + (1+y)^n$$

$$+ (1+y)^{n+1} + \dots + (1+y)^{2n} \\ = \sum_{r=0}^{2n} b_r y^r \quad [\text{Using } a_k = 1, \forall k \geq n]$$

Equating the coefficients of y^n on both sides we get

$$\Rightarrow {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \dots + {}^{2n}C_n = b_n$$

$$\Rightarrow ({}^{n+1}C_{n+1} + {}^{n+1}C_n) + {}^{n+2}C_n + \dots + {}^{2n}C_n = b_n \\ [\because {}^nC_n = {}^{n+1}C_{n+1} = 1]$$

$$\Rightarrow b_n = {}^{n+2}C_{n+1} + {}^{n+2}C_n + \dots + {}^{2n}C_n \\ [\because {}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r]$$

Combining the terms in similar way, we get

$$\Rightarrow b_n = {}^{2n}C_{n+1} + {}^{2n}C_n \Rightarrow b_n = {}^{2n+1}C_{n+1}$$

14. We know

$$(1-x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + \dots + (-1)^n C_n x^n$$

On multiplying both sides by x ,

$$x(1-x)^n = C_0x - C_1x^2 + C_2x^3 - C_3x^4 + \dots + (-1)^n C_n x^{n+1}$$

On differentiating both sides w.r. to x ,

$$(1-x)^n - nx(1-x)^{n-1} \\ = C_0 - 2C_1x + 3C_2x^2 - 4C_3x^3 + \dots + (-1)^n (n+1) C_n x^n$$

Again on multiplying both sides by x ,

$$x(1-x)^n - nx^2(1-x)^{n-1} \\ = C_0x - 2C_1x^2 + 3C_2x^3 - 4C_3x^4 + \dots + (-1)^n (n+1) C_n x^{n+1}$$

On differentiating both sides with respect to x ,

$$(1-x)^n - nx(1-x)^{n-1} - 2nx(1-x)^{n-1} + nx^2(n-1)(1-x)^{n-2} \\ = C_0 - 2^2C_1x + 3^2C_2x^2 - 4^2C_3x^3 + \dots + (-1)^n (n+1)^2 C_n x^n$$

Putting $x = 1$, in above, we get

$$0 = C_0 - 2^2C_1 + 3^2C_2 - 4^2C_3 + \dots + (-1)^n (n+1)^2 C_n$$

$$15. {}^{n+1}C_1 + {}^{n+1}C_2s_1 + {}^{n+1}C_3s_2 + \dots + {}^{n+1}C_{n+1}s_n$$

$$= \sum_{r=1}^{n+1} {}^{n+1}C_r s_{r-1},$$

$$\text{where } S_n = 1 + q + q^2 + \dots + q^n = \frac{1-q^{n+1}}{1-q}$$

$$\therefore \sum_{r=1}^{n+1} {}^{n+1}C_r \left(\frac{1-q^r}{1-q} \right) = \frac{1}{1-q} \left(\sum_{r=1}^{n+1} {}^{n+1}C_r - \sum_{r=1}^{n+1} {}^{n+1}C_r q^r \right) \\ = \frac{1}{1-q} [(1+1)^{n+1} - (1+q)^{n+1}] \\ = \frac{1}{1-q} [2^{n+1} - (1+q)^{n+1}] \quad \dots (i)$$

$$\text{Also, } S_n = 1 + \left(\frac{q+1}{2} \right) + \left(\frac{q+1}{2} \right)^2 + \dots + \left(\frac{q+1}{2} \right)^n \\ = \frac{1 - \left(\frac{q+1}{2} \right)^{n+1}}{1 - \left(\frac{q+1}{2} \right)} = \frac{2^{n+1} - (q+1)^{n+1}}{2^n(1-q)} \quad \dots(ii)$$

From equations (i) and (ii),

$${}^{n+1}C_1 + {}^{n+1}C_2s_1 + {}^{n+1}C_3s_2 + \dots + {}^{n+1}C_{n+1}s_n = 2^n S_n$$

$$16. S = \sum \sum C_i C_j$$

$$0 \leq i < j \leq n$$

$$\Rightarrow S = C_0(C_1 + C_2 + C_3 + \dots + C_n) + C_1(C_2 + C_3 + \dots + C_n) \\ + C_2(C_3 + C_4 + C_5 + \dots + C_n) + \dots + C_{n-1}(C_n) \\ \Rightarrow S = C_0(2^n - C_0) + C_1(2^n - C_0 - C_1) + C_2(2^n - C_0 - C_1 - C_2) \\ + \dots + C_{n-1}(2^n - C_0 - C_1 - \dots - C_{n-1}) + C_n(2^n - C_0 - C_1 - \dots - C_n) \\ \Rightarrow S = 2^n(C_0 + C_1 + C_2 + \dots + C_{n-1} + C_n) \\ - (C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2) - S$$

$$\Rightarrow 2S = 2^n \cdot 2^n - \frac{2n!}{(n!)^2} = 2^{2n} - \frac{2n!}{(n!)^2}$$

$$\Rightarrow S = 2^{2n-1} - \frac{2n!}{2(n!)^2}$$

17. Given :

$$C_1 + 2C_2x + 3C_3x^2 + \dots + 2nC_{2n}x^{2n-1} = 2n(1+x)^{2n-1} \quad \dots(i)$$

$$\text{where } C_r = \frac{2n!}{r!(2n-r)!}$$

Integrating both sides with respect to x , under the limits 0 to x , we get

$$[C_1x + C_2x^2 + C_3x^3 + \dots + C_{2n}x^{2n}]_0^x = [(1+x)^{2n}]_0^x \\ \Rightarrow C_1x + C_2x^2 + C_3x^3 + \dots + C_{2n}x^{2n} = (1+x)^{2n} - 1 \\ \Rightarrow C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_{2n}x^{2n} = (1+x)^{2n} \quad \dots(ii)$$

Changing x by $-\frac{1}{x}$, we get

$$\Rightarrow C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \frac{C_3}{x^3} + \dots + (-1)^{2n} \frac{C_{2n}}{x^{2n}} = \left(1 - \frac{1}{x}\right)^{2n}$$

$$\Rightarrow C_0x^{2n} - C_1x^{2n-1} + C_2x^{2n-2} - C_3x^{2n-3}$$

$$+ \dots + C_{2n} = (x-1)^{2n} \quad \dots(iii)$$

Multiplying eqn. (i) and (iii) and equating the coefficients of x^{2n-1} on both sides, we get

$$-C_1^2 + 2C_2^2 - 3C_3^2 + \dots + 2nC_{2n}^2 \\ = \text{coeff. of } x^{2n-1} \text{ in } 2n(x-1)(x^2-1)^{2n-1} \\ = 2n [\text{coeff. of } x^{2n-2} \text{ in } (x^2-1)^{2n-1} - \text{coeff. of } x^{2n-1} \text{ in } (x^2-1)^{2n-1}]$$

$$= 2n [{}^{2n-1}C_{n-1}(-1)^{n-1} - 0]$$

$$= (-1)^{n-1} \cdot 2n {}^{2n-1}C_{n-1}$$

$$\Rightarrow C_1^2 - 2C_2^2 + 3C_3^2 + \dots + 2nC_{2n}^2$$

$$= (-1)^{n-1} \cdot 2n {}^{2n-1}C_{n-1} = (-1)^{n-1} \cdot \left(\frac{2n}{n} \cdot {}^{2n-1}C_{n-1} \right)$$

$$= (-1)^{n-1} \cdot 2n C_n = (-1)^{n-1} \cdot n C_n \quad (\because {}^{2n}C_n = C_n)$$